Eur. Phys. J. B **11**, 361–364 (1999)

THE EUROPEAN PHYSICAL JOURNAL B
EDP Sciences

Multifractal critical phenomena in traffic and economic processes

A. Bershadskii[a]

P.O. Box 39953, Ramat-Aviv 61398, Tel-Aviv, Israel

Received 5 February 1999

Abstract. Multifractal critical phenomena with infinite-temperature critical point and with complex coexistence of the infinite and finite temperature critical points are considered and it is shown that strange attractors generated by cascades of period-doubling bifurcations (Feigenbaum scenario) as well as fields of velocity differences in fluid turbulence belong to the former subclass of the multifractal critical phenomena, while the real traffic processes and real currency exchange processes belong to the last (complex) subclass of the multifractal critical phenomena. Data obtained by different authors are used for this purpose.

PACS. 05.40.-a Fluctuation phenomena, random processes, noise, and Brownian motion – 01.75.+m Science and society

1 Introduction

Random processes generated by human activity attract a great interest of physicists and mathematicians in last years. Since 1997 we see an explosion in "econophysics". Statistical analysis of real data, numerical simulations and some analogies with other random processes are used to understand origin of chaos in the human activity induced (HAI-) processes. In present time a most popular analogy is comparison with fluid turbulence (see, for instance, [1–14] and references therein). Since the cascades of period-doubling bifurcations (Feigenbaum scenario [15]) are considered as an universal origin of the chaos in the fluid turbulence this analogy implies that the Feigenbaum scenario should play analogous role in the HAI-processes.

Comparison of the real data obtained in the HAI processes with analogous data obtained for chaos generated by simple maps (*via* the Feigenbaum scenario) as well as with data obtained in real turbulent flows is a direct way to check applicability of this analogy and its restrictions. In this paper we show that both these types of processes can be considered as multifractal critical phenomena. However, while the fields of velocity differences in fluid turbulence (as well as the strange attractors generated *via* the Feigenbaum scenario) belong to a simple subclass of the multifractal critical processes with infinite multifractal critical temperature, the real traffic and economic (currency exchange) processes belong to more complex subclass of the multifractal critical phenomena which we called as "two-branches" subclass. For the last subclass of the multifractal critical phenomena coexistence of the infinite and finite critical points takes place. Moreover, for some value of the multifractal temperature a crossover from the branch growing from the infinite-temperature critical point to the branch growing from the finite-temperature critical point (or even merging of these two branches) takes place for processes belonging to this subclass. This implies, in particular, that mathematical models of the considered here HAI-processes and mathematical models of the fluid turbulence should have principal differences.

[a] e-mail: bersh@hotmail.com

2 Critical points

Generalized scaling implies scaling relationship between moments of different order

$$F_q \sim F_p^{\rho(q,p)}. \tag{1}$$

The exponent $\rho(q,p)$ obeys the following obvious equation

$$\rho(q,p)\rho(p,q) = 1. \tag{2}$$

It follows from (2) that the function of two variables: $\rho(q,p)$, can be represented using a function of one variable only

$$\rho(q,p) = \frac{f(q)}{f(p)}, \tag{3}$$

and to find a general form of the function of one variable: $f(p)$, is main purpose of present paper.

The critical point p_c is defined from the equation

$$f(p_c) = 0. \tag{4}$$

For usual multifractality the moment's order - p, can be interpreted as an inverse temperature $p \sim 1/T$ of a virtual multifractal thermodynamics [16,17]. Therefore, it is interesting to check a phase-transition-like behavior of the function $f(p)$ in a vicinity of the critical point (*cf.* [18])

$$f(p) \sim (p - p_c)^\gamma \tag{5}$$

where γ is some critical index.

Since p is *a* dimensionless quantity one should normalize the virtual multifractal temperature T to obtain the order parameter p. If one studies the critical phenomena it seems natural to normalize T using the critical temperature T_c. At this normalization $p_c = 1$ and, therefore,

$$f(p) \sim (p - 1)^{\gamma_1} \tag{6}$$

in a vicinity of the critical temperature. There is also a possibility that the critical temperature is infinite (*i.e.* $p_c = 0$). This case corresponds to a phase transition from a state corresponding to the negative multifractal temperatures to a state corresponding to the positive multifractal temperatures (*cf.* [16,18] and for a modern advance [19]). In this case

$$f(p) \sim p^{\gamma_0}. \tag{7}$$

It should be noted that the representation (7) for $f(p)$ was suggested (and rigorously proved) for the first time for random walks probability distribution on linear fractals in [20] and numerically confirmed for the random walks probability distribution on percolation clusters [20,21] (see also [22]).

If there is an ordinary scaling

$$F_q(r) \sim r^{\zeta_q} \tag{8}$$

then

$$\frac{\zeta_q}{\zeta_p} = \frac{f(q)}{f(p)}. \tag{9}$$

3 Examples of the stochastic processes with infinite critical temperature

It is well known that dissipative dynamical systems, that exhibit the cascade of period-doubling bifurcations, are in practice well modeled by one-dimensional maps with a single quadratic extremum such as the map

$$x_{n+1} = 1 - a|x_n|^2. \tag{10}$$

As one increases the parameter a one observes an infinite sequence of subharmonic bifurcations at each stage of which the period of the limit cycle is doubled. This period-doubling cascade accumulates at $a_c = 1,40115...$ where the system possesses a 2^∞-orbit that displays scale invariance. Beyond this critical value, the attractor becomes chaotic [15]. This scenario presents strong analogy with the second order phase transition (see [17] and references therein). Analogous scenario (generally called Feigenbaum scenario) takes also place for other values of $z > 2$, *i.e.* for generalized map

$$x_{n+1} = 1 - a|x_n|^z. \tag{11}$$

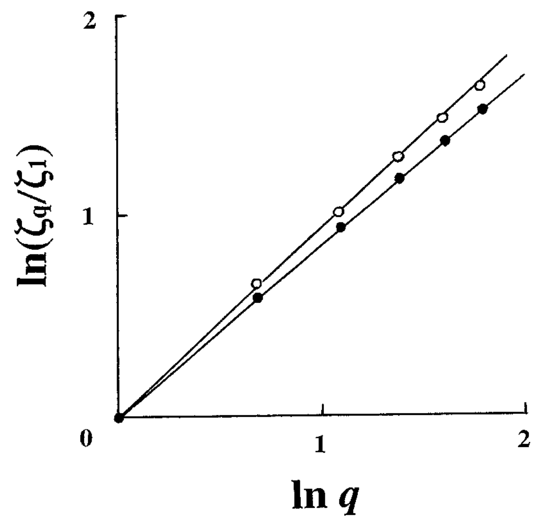

Fig. 1. $\ln(\zeta_q/\zeta_1)$ against $\ln(q)$ for the critical chaos generated by the map $x_{n+1} = 1 - a|x_n|^z$ (full circles correspond to $z = 3$ and open circles correspond to $z = 2$). Data are taken from [24]. Straight lines are drawn to indicate agreement with the critical representation $\zeta_q \sim q^{\gamma_0}$.

In paper [23] the generalized dimension spectrum D_p was computed for different critical strange sets corresponding to (11). Then, in a more recent paper [24] the existence of a global universality for the generalized dimensions D_p on all critical points of phase transitions from period-η-tuplings to chaos in $1D$ iterative systems was established, that gives a possibility to consider all these multifractal spectra (for given map) in an universal form, introducing normalized generalized dimensions D_p/D_0. In the general case one can define the scaling exponents ζ_p using the generalized dimensions

$$\zeta_p = D_p(p - 1) + D_0 \tag{12}$$

then

$$\frac{\zeta_p}{\zeta_1} = \frac{D_p}{D_0}(p - 1) + 1. \tag{13}$$

In Figure 1 we show these universal scaling data (taken from [24]) for $z = 2$ (open circles) and for $z = 3$ (full circles). The log-log axes: $\ln(\zeta_q/\zeta_1)$ and $\ln q$, are chosen for comparison with the critical representation: $\zeta_q \sim q^{\gamma_0}$ (corresponding to the infinite critical temperature) and the straight lines are drawn to indicate good agreement between the critical representation and the numerical data. One can also extract values of the critical exponent γ_0 from this figure: $\gamma_0 \simeq 0.93$ for $z = 2$ and $\gamma_0 \simeq 0.86$ for $z = 3$.

In the last years the statistic properties of the traffic and economic processes have been compared, as a rule, with analogous properties of velocity differences in turbulence [1,2]. Let us consider the corresponding moments of the velocity differences at the scales r

$$F_q = \langle |v(x + r) - v(x)|^q \rangle \sim r^{\zeta_q}. \tag{14}$$

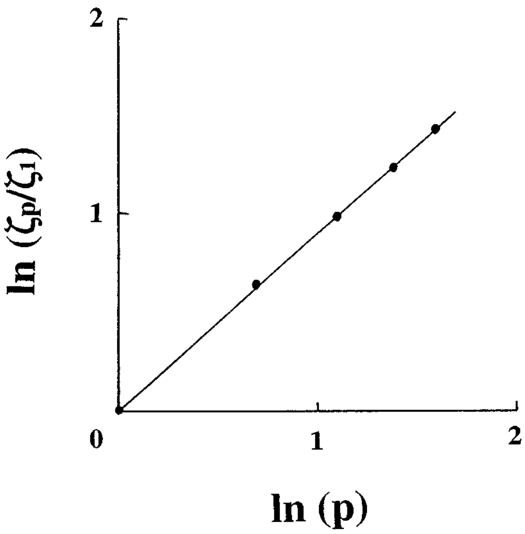

Fig. 2. $\ln(\zeta_p/\zeta_1)$ against $\ln(p)$ for the data represented in [25] and obtained in different turbulent flows for the velocity differences field.

A particular case of the generalized scaling of the turbulent processes then can be represented as

$$F_q \sim F_1^{\zeta_q/\zeta_1} \tag{15}$$

(see, for instance, [25,26] and references therein). A remarkable property of the generalized multifractality is that the generalized scaling (15) can be observed for low Reynolds numbers or for small scales belonging to the dissipative range where usual scaling is not verified, but is expected for large values of the Reynolds number [25].

The data shown in Figure 2 correspond to fully developed turbulence, to thermal convection and to magnetohydrodynamic turbulence (the data taken from [25]). All these processes have very close values of ζ_q/ζ_1. One can see that these data can be well fitted by the approximation (7, 9) (corresponding to the infinite critical temperature) with the critical index $\gamma_0 \simeq 0.9$. It is interesting to compare this value of the critical index with the values obtained above for the Feigenbaum chaos. It seems rather plausible that for some value of z, belonging to the interval $[2, 3]$, the generalized map (11) should generate Feigenbaum attractor with the critical index $\gamma_0 = 0.9$ (due to $\gamma_0 \simeq 0.93$ for $z = 2$ and $\gamma_0 = 0.86$ for $z = 3$). In any case, it is clear that the multifractality of the turbulent velocity differences belongs to the same subclass that the multifractality of the Feigenbaum strange attractors generated by the generalized map (11) do, *i.e.* to the subclass with *infinite* multifractal critical temperature.

4 Two-branch critical phenomena with finite critical temperature

Multifractality of the real traffic and economic processes, however, seems to be more complex than multifractality of the above considered Feigenbaum chaos and stochastic field of velocity differences in fluid turbulence. If the system has both the finite and infinite multifractal critical points it is nontrivial to chose from (9) what branch of the f-function (corresponding to $p_c = 0$ or corresponding to $p_c = 1$) determines the ordinary scaling exponent ζ_p for $p > 1$. In this case, it is possible that ζ_p can cross over between these two critical branches or even these branches can merge for some $p_{\rm mer} > 1$. After this crossover (or merging), *i.e.* for $p > p_{\rm mer}$

$$\zeta_p \sim (p-1)^{\gamma_1}. \tag{16}$$

Let us show that just this complex (two-branches) case takes place in real traffic and economic data.

In paper [27] intermittent behavior of traffic flow was related to switches between a congested flow regime and a free flow regime. $1/f$-noise corresponds to such model intermittency (see also [28]). On the other hand, in a recent paper [2] a reliable statistical analysis of a large amount of high-resolution traffic data, with the velocities and arrival times recorded of any car that passed an induction loop detector, was performed and $1/f^\alpha$-noise (with $\alpha \simeq 1.14$) was found. Moreover, the authors of reference [2] report also results of a high order statistical analysis of these data: high order moments of the velocity differences $\Delta v_\tau = v(t-\tau) - v(t)$, between cars a time-delay τ apart. This analysis show scaling behavior of the moments

$$F_p(\tau) = \langle (\Delta v(t-\tau) - v(t))^p \rangle \sim \tau^{\zeta_p}. \tag{17}$$

In Figure 3 we show a set of ζ_p (full circles) extracted in [2] from the data of velocities of single cars crossing an induction loop detector, collected at Köln-Nord over more than one week. The raw data set contains a total of 515, 429 data-points. We use only even (see [25]) moments with $p = 2, 4, 6, 8, 10$. The log-log axes are chosen for comparison with the critical representation (16). The straight line indicates good agreement between the data and the critical representation (16).

Let us now consider some real stochastic processes in economics. In a recent paper [9] a multi-affine analysis of typical currency exchange rates was performed. The moments were defined as

$$F_p(\tau) = \langle |y(t) - y(t')|^p \rangle_\tau \tag{18}$$

with $\tau = |t - t'|$, the currency exchange rates $y(t)$ were taken as the closing values of successive open banking days recorded in Brussels market for the USD/DEM and JPY/USD exchange rates from Jan. 1, 1980 to Dec. 31, 1996. Only non-zero terms were taken into account in the average over all couples of points $(y(t), y(t'))$. About 4400 points were considered for each currency in the period covering 16 years. The scaling law

$$F_p(\tau) \sim \tau^{\zeta_p} \tag{19}$$

was observed for the moments (18). The exponents ζ_p extracted in [9] from these data are also shown in Figure 3 (open circles). It should be noted that ζ_p/ζ_2 are approximately the same for USD/DEM and for JPY/USD data. The straight line indicate agreement of the data with the critical representation (16).

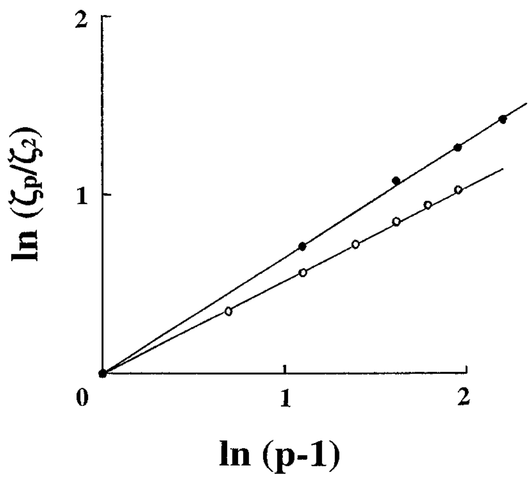

Fig. 3. $\ln(\zeta_p/\zeta_2)$ against $\ln(p-1)$ for the data (full circles) collected at Köln-Nord over more than one week [2] for velocities of single cars crossing an induction loop detector and for the typical currency exchange rates taken as the closing values of successive open banking days recorded in Brussels market for the USD/DEM and JPY/USD exchange rates from Jan. 1, 1980 to Dec. 31, 1996 (data - open circles, are taken from [9]). Straight lines are drawn to indicate agreement with the critical representation (16).

5 Discussion

It seems to be expectable that the stochastic fields of the velocity differences in fluid turbulence belong to the same multifractal subclass as the strange attractors generated by the cascade of period-doubling bifurcations do (these cascades were very widely discussed in the literature as an origin of chaos in fluid turbulence velocity fields). However, the above considered traffic and economic processes belong to the more complex (two-branches) class of the critical multifractal processes (it should be noted, that the fits get much more worse if we use p instead of $(p-1)$ as a variable for the traffic and economic data, and $(p-1)$ instead of p for the "cascade" data). Therefore, the analogy between the traffic and economic random processes and the fields of turbulent velocity differences has rather restricted character. This observation does not still mean that mathematical models of the random processes generated by human activity should be more complex than the hydrodynamical models are. It just means that these models should have principal differences and, in particular, that origin of the chaos in the traffic and economic processes is not related directly to the cascades of period-doubling bifurcations.

The author is grateful to S. Lovejoy and to F. Schmitt for sending reprints of their papers, and to D. Stauffer and to an anonymous Referee for information, comments and suggestions.

References

1. R.N Mantegna, H.E. Stanley, Nature **376**, 46 (1995).
2. P. Wagner, J. Peinke, Z. Naturforsch. A **52**, 600 (1997).
3. J.C. Vassilicos, A. Demos, F. Tata, in *Applications of fractals and chaos*, edited by A.J. Crilly *et al.* (Berlin, Springer-Verlag, 1993), p. 249.
4. F. Schmitt, D. Schertzer, S. Lovejoy, in *Chaos, fractals and models 96*, edited by F.M. Guindani, S. Salvadori (Italian University Press, Roma, 1998), p. 150.
5. F. Schmitt, D. Schertser, S. Lovejoy, *Multifractal analysis of foreign exchange data*, Appl. Stoch. Models Data Analys. (in press).
6. F. Schmitt, D. Schertser, S. Lovejoy, *Turbulent fluctuations in financial markets: a multifractal approach* (preprint).
7. A. Fisher, L. Calvet, B. Mandelbrot, Cowles Foundation Discussions Paper 1166 (1991).
8. N. Vandewalle, M. Ausloos, Int. J. Mod. Phys. C **9**, 711 (1998).
9. N. Vandewalle, M. Ausloos, Eur. Phys. J. B **4**, 257 (1998).
10. K. Ivanova, M. Ausloos, Eur. Phys. J. B **8**, 665 (1999).
11. N. Vandewalle, M. Ausloos, Physica A **246**, 454 (1997).
12. A. Arneodo, J.-F. Muzy, D. Sornette, Eur. Phys. J. B **2**, 277 (1998); stock market from the "infrared" to the "ultraviolet", `cond-mat/9708012`.
13. S. Ghashghaie *et al.*, Nature **381**, 767 (1996).
14. A. Hilgers, C. Beck, Int. J. Bifurc. Chaos **7**, 1855 (1997).
15. M.J. Feigenbaum, J. Stat. Phys. **19**, 25 (1978); **21**, 669 (1979); **46**, 919 (1987); **52**, 527 (1988).
16. H.E. Stanley, P. Meakin, Nature **335**, 405 (1988).
17. A. Arneodo, E. Bacry, J.F. Muzy, Physica A **213**, 232 (1995).
18. L.D. Landau, E.M. Lifshitz, *Statistical Physics*, part 1 (Pergamon Press, 1980).
19. A. Vourdas, R.F. Bishop, J. Phys. A **31**, 8563 (1998).
20. S. Havlin, A. Bunde, Physica D **38**, 184 (1989).
21. A. Bunde, S. Havlin, H.E. Roman, Phys. Rev. A. **42**, 6274 (1990).
22. A. Bershadskii, J. Stat. Phys. **87**, 607 (1997).
23. B.-L. Hao, W.-Z. Zeng, Chinese Phys. Lett. **3**, 285 (1986).
24. K.-F. Cao, S.-L. Peng, J. Phys. A **25**, 589 (1992).
25. R. Benzi *et al.*, Physica D **96**, 162 (1996).
26. A. Bershadskii, J. Phys. A **29**, L453 (1996).
27. K. Nagel, M. Paczuski, Phys. Rev. E **51**, 2909 (1995).
28. K. Nagel, H.J. Herrmann, Physica A **199**, 254 (1993).